A Real Paradox?

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Here is a recent email exchange between myself and NASA researcher Michael Holloway:

> -------- Forwarded Message --------
> Subject: ....
> Date: Thu, 15 Jan 2015 20:04:44 +0100
> From: Peter Bernard Ladkin <ladkin@rvs.uni-bielefeld.de>
> To: Holloway, Michael (LARC-D320) <c.michael.holloway@nasa.gov>
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> On 2015-01-15 18:59 , C. Michael Holloway (LaRC-D320) wrote:
>> Every person believes things that are nonsense.
>
> Nonsense. Although thereby demonstrable for you, I personally believe nothing that is nonsense.
> Except for the previous sentence.

My response is not as rude as it may seem. I spotted what could be a logical paradox.

Logic is the study of inferences between statements. The study of logic is the study of what inferences are legitimate and why. Modern logic has progressed mostly in the study of inferences in mathematics, for historical reasons (see below) and most modern logicians work in this area. But the study of inferences amongst sentences of everyday discussion was where I discovered logic when I was 14 (I think it was). I read Basson and O'Connor's book Introduction to Symbolic Logic on the formal logic of statements, which is called propositional logic, and thought: Wonderful! - there is a formal way to tell whether people's reasoning in discussion is correct or not! Because it used to bother me as an argumentative teenager that people could contradict themselves in discussion and declined to acknowledge that they were making no sense.

Real life unfortunately doesn't turn out to be quite that simple. You have to get people to acknowledge that they are contradicting themselves, and then that contradiction invalidates their point of view, and neither of those two tasks is usually simple. After all, most people who contradict themselves either haven't noticed or don't care.

Take the following contemporary discussion. For a communication to be "private" or "confidential" between A and B means that only A and B know what was said, and no one else. Communications between lawyers and clients in the UK are held to be confidential, and indeed the confidentiality of such communications is a principle of British law. But the current UK Prime Minister has just announced that he wishes it to be possible for any electronic communications (such as over the Internet, as well as phone calls) to be decrypted as "necessary" by British intelligence services. Such a requirement, if enshrined in law, would contradict other legal principles. If communication between A and B is to be confidential, then by the above meaning of the word no third party, such as an agent of a British intelligence service, can know what was said. And if an agent of an intelligence service is enabled access (via whatever legal safeguards) to any electronic communication between A and B, then that communication cannot be assuredly confidential.

That's what logic yields. It follows from this that whether a communication remains confidential depends (in this case, solely) on the whim of the intelligence services. But many people engaged in the discussion of whether that is a good idea or not have not seen the contradiction which needs to be resolved: the proposed measure of enabling access to communications means ipso facto that such communications cannot be assured to be confidential. Assured confidentiality is held to be crucially important for the correct operation of British law by lawyers and their clients, and judges. One "side", either the legleagles or the politicians, is bound to lose that argument. Someone has to give up what is, for them, a major principle. They can't be "reasonable", shake hands, and "agree". It's not like the case in which one party X wants the bus to run every ten minutes and another party Y wants to run it every hour and X and Y can agree to run the bus every half hour. That would be a compromise. There is no compromise with contradictory principles (another gem from logic). One principle, either the legal principle of assured confidentiality, or the (to me and others, questionable) intelligence principle of access to any communication as needed, must be given up here. Anyone who says "let's find a middle ground" is pulling the wool over someone's eyes: there is none.

So logic can help in clarifying starkly what needs to be resolved. At least, as a teenager I thought it could. Half a century later I'm jaded but haven't given up hope.

So let's get back to the interchange between Michael and me. The thing is this. When you read the sentences they make sense and, as my son Simon observed, Michael could well be right in terms of wisdom about the everyday world and people in it.

But when you try to formulate it in formal logic it is hard to make it work. Such things are called "logical paradoxes".

Let's take as a prime example of "nonsense" a contradiction: (X AND NOT-X). Somebody says "two plus two is four and two plus two is not four". You'd think, either it is four or it isn't four; it can't be both (and you'd be right), so our stater is asserting nonsense.

Michael says: FOR EVERY person P, THERE IS a statement X, SUCH THAT P believes (X AND NOT-X).

I say: this is NONSENSE, but what I really mean is this is not true, that is
NOT (FOR EVERY person P, THERE IS a statement X, SUCH THAT P believes (X AND NOT-X)).

Using generally-agreed principles of logic, we can "take" the NOT "inside" the sentence:
THERE IS a person P SUCH THAT NOT(THERE IS a statement X, such that (P believes (X AND NOT-X))

and pushing the NOT "inside" even further:

THERE IS a person P SUCH THAT FOR EVERY statement X, it is NOT the case that (P believes (X AND NOT-X))

Further, I suggest that an example of such a person is me.

If it's not the case that P believes (X AND NOT-X), can we infer from this that P must believe NOT (X AND NOT-X)?

There are lots of statements I cannot comprehend – lots of statements concerning particularly involved and abstruse areas of mathematics, for example. Let's take one, call it Y. I don't understand Y, cannot fathom it as it stands. Wouldn't it be hard to say I *believe* NOT (Y AND NOT

Y) if I can’t even figure out what Y says. How could I be said to really believe something I can’t understand?

There is a way: maybe I apply “higher principles”, say that of classical logic. Consider the “principle of non-contradiction”, which in one formulation says that, no matter what X may be, (X AND NOT-X) can never be an acceptable assertion. If I believe the principle of non-contradiction, then I am going to believe NOT (X AND NOT X) no matter what X may be. So in particular it goes for the to-me-incomprehensible Y. I don’t understand Y, but if I hold the principle of non-contradiction, it must also apply to Y even though I don’t understand it.

But do I really hold the principle of non-contradiction? Consider the following.

I’ve written a book about analysing engineered systems causally. You can read it at http://www.rvs.uni-bielefeld.de/publications/books/CausalSystemAnalysis/index.html . I tried hard to get it all right. Everything. Ignoring the table of contents, index and bibliography, the book consists of a set of sentences/statements A1, A2, A3, ... Ak for some large number k. I don’t know how many - some 10,000 or so. Seen as one large logical statement, the book asserts (A1 AND A2 AND A3 AND .... AND Ak). Say I write a preface in which I say "*I have tried very hard to ensure everything I wrote is correct, and thank my proofreaders...... for correcting some errors. Nevertheless, I am sure some errors remain.*" It is quite usual to write something like that. But if I think some error remains, that must mean that I think that at least one of A1, A2,....... is in fact incorrect, that is, is false, although I don't know which. Enumerating them, I have asserted (NOT A1) OR (NOT A2) OR (NOT A3) OR.....OR (NOT Ak) in the preface. Which is logically equivalent to NOT (A1 AND A2 AND A3 AND ...... AND Ak).

I believe both (A1 & A2 AND A3 AND .... AND Ak) and NOT (A1 AND A2 AND A3 AND ...... AND Ak). Letting Y denote (A1 AND A2 AND A3 AND ...... AND Ak), I believe Y and also NOT-Y. If I believe two things X and Z, I surely must believe their conjunction (X AND Z). So it seems I believe Y AND NOT-Y. Furthermore, it’s reasonable for me to believe it - I do believe everything I wrote, including the statement that I nevertheless probably made a mistake somewhere.

So I seems that I don’t live by the principle of non-contradiction in my everyday reasoning. (That example, by the way, is known as the Paradox of the Preface.)

In classical logic there is an inference rule, traditionally called "*ex falso quodlibet*", which says that from a contradiction you can infer any statement. That is, suppose I assert A AND NOT-A. Then I may infer any statement B whatsoever from this. In logic, we usually write this as

$$\frac{A \; \& \; (\text{NOT } A)}{B}$$

where A and B are arbitrary statements (full sentences that say something either true or false). What is above the line is what you “have” / accept / assert / have derived already, and what is below the line is the statement which you may infer from what is above the line. It is called a “rule” because both statements above and below the line are schematic – A and B may be replaced by arbitrary statements.

So, since I believe the Preface contradiction, if I use *ex falso quodlibet* it seems I should believe (rather: be committed to believing) everything whatsoever. Which I do not do (as far as I can tell), and neither can I imagine doing so.

So it seems as if two principles of “classical logic”, the principle of non-contradiction and the inference rule *ex falso quodlibet*, just aren’t valid in logical reasoning about “everyday life”, whatever that is.

So what use is “classical logic”, then? In particular, since it is used pervasively in mathematics and engineering, in particular in using mathematical logic to show that critical computer programs correctly do what they are specified to do, shouldn’t we be somewhat sceptical that it works?

Maybe it’s OK, though, if we stick to math. The "standard" story is that Frege solved the problems of medieval logic (which involved the semantic interactions of multiple "quantifiers". A quantifier is a non-object-specific noun-like term such as "nobody", "everybody", "somebody"), essentially through "scoping" , namely, giving a specific order in which you interpret the quantifiers. So, for example, "*everybody loves somebody*" can be read "*for every person, there is some person whom heshe loves*" or "*there is some (one) person whom everybody loves*" and these two readings are different statements. Frege said that all you need are the quantifiers "*for everything x*" and "*for something x*" where x is a placeholder, a pronoun if you like, which occurs explicitly somewhere in what follows in the rest of the statement. If you build sentences exclusively out of these quantifiers, then they are unambiguous and the meaning is straightforward to determine from the meanings of the statement parts, showed Frege. And it turns out that, with very few exceptions, indeed most statements of ordinary language can be put in Fregean-quantificational form - the great twentieth-century philosopher W.V.O Quine held it as a cornerstone of his philosophy that all meaningful statements can be so written. Frege proposed his ideas publicly first in the *Begriffschrift*, a booklet published in 1879. His logic is written in a cute two-dimensional visual notation – nowadays we make use of a few extra symbols in the line. Gödel proved formally in 1930 that this formal logic suffices for Frege’s purposes.

Frege’s explicit purpose was to encode the logical truths and inferences in mathematics, as he attempted in the *Grundgesetze der Arithmetic*, of 1893. Frege thought that mathematics was just pure logic, pure reasoning, but his attempt to derive all of math in logic foundered when Bertrand Russell discovered "Russell's Paradox", a contradiction in Frege's theory of sets, in 1901. Frege gave up in despair but his work is the foundation of all modern logic, classical and non-classical, and Russell's Paradox is the first of a series of genuinely new issues, possibly the first for five hundred years, which have kept people busy since then trying to sort them out.

It is said (amongst others by Ernie Adams who was one of my thesis advisors) that the four greatest logicians (students of correct inference) were Aristotle, Frege, Gödel and Tarski. Gödel was a recluse, starting in Vienna and ending up in Princeton, New Jersey. He was reported to have had some odd paranoias. It is interesting that someone supposedly so interested in correct reasoning should at the same time be subject to such irrational beliefs. That, though, is part of what Michael suggested, is it not? Tarski was in Berkeley, retired when I got there but I still got to talk with him once or twice. I have a couple of photos taken at his 80th birthday bash in 1981, which is where Ernie made the above statement.

Tarski stories were valuable currency at Berkeley in those days. I have mine. Here it is for what it’s worth. The Math Department was visited by the Russian group theorist S. I. Adjan to give a Logic Colloquium (held every second Friday afternoon when I was there). I seem to remember it had something to do with the Burnside Problem (see, for example, the entry in the CRC Concise Encyclopedia of Mathematics, available on-line through Google). Adjan had recently been through a theorem of Thompson (I think it was), who had written a very-detailed paper of a few hundred pages settling a long-standing conjecture in group theory. (It could well have been the 254pp Feit-Thompson paper of 1963 in the Pacific Journal of Mathematics. It would be appropriate for a Berkeley talk, since the PJM is published out of Berkeley.) It was regarded as an astonishing tour de force, for these kinds of calculations are not easy, let alone pursuing them for hundreds of pages. The average math graduate student would be content with a dozen of them for a thesis in group theory. I'd have been happy just being able to do a few lines, given my meagre skills in group-theoretic calculation. Adjan had gone through carefully and found a "mistake" at the bottom of, say, page 49 (this number sticks in my memory, but that's no guarantee of anything). And he proved it was a mistake – it was a condition which he showed couldn't be realised and Feit-Thompson, if indeed it was they, had thought it could. Adjan had fixed it and resolved the problem ("fixing" has consequences and you have to work those through, which can often be harder than the original work. As happened, for example, with the eventual proof of Fermat's Last Theorem).

Adjan's work was an impressive piece of mathematics involving astonishing complexity and of course Tarski was there to hear about it. So, rather than the usual drinks-and-snacks after the Colloquium, a dinner had been organised at Tarski's favorite upscale restaurant (his other favorite seems to have been a pizza parlor which was open late and in which Maria and he could occasionally be seen close to midnight). I signed up, as I usually did for the social events. We all got to the Metropole, as it was called; and were ushered to a long table in the basement. I went to the toilet. When I came back, there was just one seat available, at the end, next to the genius Adjan and opposite the genius Tarski, with Maria sitting next to him. Gulp. Fright. What do I do? Go to Ralph or Roger and ask them if they'd swap seats? After all, they both knew Alfred and Maria already. But then, if that would work, why weren't they sitting there already? Maybe they're just as scared as me? I'm a tough guy, I can tough it out. I can sit down right there - and grin sheepishly and whimper just like anybody else would. What on earth could I say to such unique people that would be at all interesting?

In fact, the dinner was an object lesson in social courtesy. I couldn't remember having a better time anywhere. Someone was always talking with me in English, while the other two were talking in Russian (in which Tarski and Maria were both fluent). And not just about general matters, but about logic and formal logic also. Talking about logic with the guy who had formalised half the basics. One of four logical greats in the history of the universe. Wow.

That's my Tarski Tale.

Gödel proved Frege's logic, first order logic, to be complete, in 1930, as I mentioned. "Complete" means that the inference rules are adequate to prove everything you can express in the logic that turns out to be always true (that is, true no matter how you interpret the variable parts, also called logical truth). Tarski defined formally what it means for a sentence in the formal language of logic to be true, . Gödel went on to prove his "Incompleteness Theorem" (there are actually two), which says that in Frege's formal arithmetic (called "Peano Arithmetic" after Guiseppe Peano who formulated the axioms, which in Frege's system are combined with his first order logic) it is not possible to prove everything that is true, and furthermore you can't fix this by adding new principles and extending the logic.

Tarski and Gödel are both known for the work they did with classical logic, Fregean logic. Indeed, Gödel did also look at some features of a particular form of what is called constructive logic, which is a logic used for so-called constructive mathematics, and he proved some fundamental theorems about it.

So that's enough history for now. Back to logic.

First order classical logic, and propositional classical logic (which considers just whole statements and doesn't employ sentence deconstruction with quantifiers) both employ the principle of non-contradiction and *ex falso quodlibet*. Which we have seen aren't necessarily very helpful for everyday logical reasoning, for reasoning about actual human beliefs and arguments.

There are other encumbrances. For example, the model theory (also invented by Tarski) of first order classical logic interprets the quantifier (FOR ALL things X) as incorporating an existence assumption: from (FOR ALL X <something about X>) you can infer (THERE EXISTS an X <same something about X>). This makes it hard to use this logic to reason about stories, about fiction, because for example the people in fiction don't exist, or are not usually taken to exist. Equally, it makes it hard for this logic to be used to talk about historical figures, who don't exist any more. Indeed, Russell once had a theory (the so-called theory of definitive descriptions) which said that if you assert anything about an object or person which doesn't exist, then what you assert is necessarily false. If that were really true, one wonders how any historians ever got a Fellowship at his Cambridge college, since why would anybody offer an academic job to people whose every assertion was false? Or, suppose as I write that London is obliterated by an atomic bomb. Then if I were to tell you about my favorite places to visit in London right here, you they would be all false. But I wouldn't know it, because I haven't yet heard that London no longer exists. But why can't I say coherent, correct-if-it-still-exists things about London which have a definite intuitive meaning, without having the meaning of what I say contingent upon some remote event which happens without my knowledge and which has nothing to do with the National Gallery, the IET, or the Apple store in Covent Garden, except in the matter of their continued existence or not?

So, finally back to Michael's and my interchange.

I say Michael's assertion is NONSENSE. Since I am interpreting NONSENSE as a contradiction, I am saying Michael's assertion is a contradiction. That is not quite true as it stands – it doesn't have the form (X AND NOT-X). But what I am sure he understood me to mean is that, in conjunction with principles of classical logic, a contradiction may be derived. And that is indeed so under certain principles of classical logic along with plausible properties of belief. If you take the principle of non-contradiction to be true, then you can well argue that no one can believe a contradiction. They may have some kind of attitude towards a contradiction, but, whatever that attitude is, it can't be belief. A psychologist may meet someone who claims to believe 2+2=5, but must infer that, yes, this person certainly likes to declaim 2+2=5, but *believe* it he can't. In the interchange, I could be taken to be putting myself forward as such an adherent of this principle.

But not really. Because in the next sentence I contradict Michael flatly, by asserting a counterexample, namely myself. This, by the way, points to one of the differences between classical logic and constructive logic. In classical logic, you can infer THERE IS something X such that <something about X> without having to produce such an X. In constructive logic, you can only make an assertion THERE IS something X such that <something about X> if you can actually exhibit such an X such that <something about X>. People who think that mathematics is all about complex mental constructions are inclined to constructive logic in principle. But rarely in practice, because it turns out to be very hard to use.

So, in fact, my remark NONSENSE can be taken to mean simply that a contradiction follows from Michael's claim. And then I exhibit how, namely by giving a putative counterexample, namely me.

But then I undermine this very assertion by saying that the statement is an exception to itself. I am indirectly calling it NONSENSE.

Things follow from this, which was my intention. I'm not going to ruin the story – I leave those consequences as an exercise for the reader. Shaggy dogs live!